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A Remark on Best Approximation of Alternating Series

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1. INTRODUCTION

Let $1 \leqslant q < \infty$ and $(1/p) + (1/q) = 1$. By S_q we denote the set of alternating series, $\sum_{n=0}^{\infty} (-1)^n a_n$, where $a_n = \int_0^1 t^n \psi(t)\, dt$ and $\|\psi\|_q \leqslant 1$, and we put $s_n = \sum_{k=0}^{n} (-1)^k a_k$. The elements of S_q are convergent.

Let $\gamma_n(-t) = \gamma_{n,p}(-t)$ denote the element of P_n that minimizes $\| p(t) - (1+t)^{-1} \|_p$, when $p(t)$ runs through P_n, the set of all polynomials of degree not exceeding n, and put

$$\beta_n(t) = \beta_{n,p}(t) = (\gamma_n(0) - (1-t)\, \gamma_n(t))/t = \sum_{k=0}^{n} b_{nk}^{(p)} t^k.$$

Then the linear combination of the partial sums $s_0, s_1, \ldots, s_n$ which (with regard to S_q) gives the best approximation to the sum $\sum_{n=0}^{\infty} (-1)^n a_n$ is given by $\sum_{k=0}^{n} b_{nk}^{(p)} s_k$ and

$$\varepsilon_n^{(p)} = \| \gamma_n(-t) - (1+t)^{-1} \|_p$$

gives the maximal error. (For details and the relation with known results on the problem of accelerating the convergence of alternating series see [4].) Jurkat and Shawyer [4] determined he error $\varepsilon_n^{(p)}$ and the matrix $(b_{nk}^{(p)})$ for the cases $p = 2$ and $p = \infty$. In the present note we shall calculate $\varepsilon_n^{(1)}$ and $(b_{nk}^{(1)})$.

The matrix $(b_{nk}^{(1)})$ turns out to be a regular and positive triangular matrix, whose entries are rational numbers. The error $\varepsilon_n^{(1)}$ is greater than and asymptotic to $4\lambda^{n+2}$ with $\lambda = 3 - 8^{1/2}$. Since $\varepsilon_n^{(p)}$ is monotonic in p and since $\varepsilon_n^{(\infty)} = \lambda^n/4$ (see [4]), we obtain

$$\lambda^n/9 \leqslant \varepsilon_n^{(p)} \leqslant \lambda^n/4 \qquad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad 1 \leqslant p \leqslant \infty.$$

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2. THE ERROR $\varepsilon_n^{(1)}$

We have $\|\gamma_n(-t)-(1+t)^{-1}\|_1=\int_{-1}^{1}|\gamma_n(x-1)/2-(3-x)^{-1}|\,dx$ and therefore

$$\varepsilon_n^{(1)}=\int_{-1}^{1}|p_n(x)-(x-3)^{-1}|\,dx \qquad \text{and}$$

$$\beta_{n,1}(t)=2\{(1-t)\,p_n(2t+1)-p_n(1)\}/t,$$

where $p_n(x)$ denotes the element of P_n which minimizes $\int_{-1}^{1}|p(x)-(x-3)^{-1}|\,dx$. Since this minimum is well known (see [2, p. 290]) we obtain

$$\varepsilon_n^{(1)}=2\{\log(1+\lambda^{n+2})-\log(1-\lambda^{n+2})\}\sim 4\lambda^{n+2}.$$

3. THE MATRIX $b_{nk}^{(1)}$

According to a theorem of Markoff (see [2, p. 82]) $p_n(x)$ is that polynomial of P_n that interpolates $(x-3)^{-1}$ at the $(n+1)$ zeros of $U_{n+1}(x)$, where $U_{n+1}(x)$ denotes the $(n+1)$th Chebyshev polynomial of the second kind. We deduce

$$\begin{aligned}(x-3)\,p_n(x)&=(x-3)\,U_{n+1}(x)\sum_{j=0}^{n}\{(x-x_j)(x_j-3)\,U'_{n+1}(x_j)\}^{-1}\\&=U_{n+1}(x)\sum_{j=0}^{n}\{(3-x_j)^{-1}-(x-x_j)^{-1}\}/U'_{n+1}(x_j)\\&=1-U_{n+1}(x)/U_{n+1}(3).\end{aligned}$$

Using (see (27) on p. 186 and (3) on p. 174 of [3])

$$U_{n+1}^{(k)}(1)=2^k\cdot k!\binom{n+k+2}{n-k+1} \qquad \text{for} \quad 0\leqslant k\leqslant n+1,$$

we obtain

$$\begin{aligned}U_{n+1}(3)\,\beta_{n,1}(t)&=\{U_{n+1}(2t+1)-U_{n+1}(1)\}/t\\&=\sum_{k=0}^{n}4^{k+1}\binom{n+k+3}{n-k}t^k\end{aligned}$$

and thus

$$U_{n+1}(3) \cdot b_{nk}^{(1)} = 4^{k+1} \binom{n+k+3}{n-k}.$$

The $b_{nk}^{(1)}$ are rational numbers, since the recurrence relation $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ (see [1, p. 782]) implies that $U_{n+1}(3)$ is an integer. By Toeplitz's theorem $(b_{nk}^{(1)})$ is a regular matrix, since

$$U_{n+1}(1) = (n+2) \qquad \text{and} \qquad U_{n+1}(3) = \{4 \cdot 2^{1/2} \cdot \lambda^{(n+2)}\}^{-1} \qquad (*)$$

and therefore

$$\sum_{k=0}^{n} b_{nk}^{(1)} = \beta_{n,1}(1) = 1 - U_{n+1}(1)/U_{n+1}(3) \to 1 \qquad \text{for} \quad n \to \infty$$

and

$$b_{nk}^{(1)} \to 0 \qquad \text{for} \quad k \to \infty \text{ and any fixed } n.$$

Using (*) and Stirling's formula for $n!$ we obtain (similarly to [4]) the following asymptotic behaviour for the $b_{nk}^{(1)}$:

If $c = 8^{1/2}$ and $\delta = 2^{1/2} - 13/8$, then $k = n \cdot 2^{1/2} + \delta + h$ and $|h| \leqslant n/4$ imply

$$b_{nk}^{(1)} = (c/\pi n)^{1/2} \exp(-ch^2/n)(1 + O(1/n + h^3/n^2)).$$

References

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